Chapter 8. Quadratic Equations and Functions

8.1. Solve Quadratic Equations

KYOTE Standards: CR 20; CA 11

In this section, we discuss solving quadratic equations by factoring, by using the square root property and by using the quadratic formula. We begin by defining a quadratic equation and what it means for a number to be a solution, or root, of the quadratic equation.

Definition 1. A quadratic equation in one variable is an equation that can be written in the form \( ax^2 + bx + c = 0 \), where \( x \) is the variable, \( a \), \( b \) and \( c \) are real numbers and \( a \neq 0 \). A solution, or root, of this equation is a value of \( x \) that makes \( ax^2 + bx + c = 0 \) a true statement. Such a number is said to satisfy the equation.

Factoring

The key to solving quadratic equations by factoring depends upon the zero-product property of real numbers.

Zero-Product Property

If \( a \) and \( b \) are real numbers such that \( ab = 0 \), then either \( a = 0 \) or \( b = 0 \).

Example 1. Solve \( 2x^2 + x - 15 = 0 \) by factoring.

Solution. If we can factor the quadratic polynomial \( 2x^2 + x - 15 \), then it is fairly easy to find its roots using the zero-product property. We obtain

\[
\begin{align*}
2x^2 + x - 15 &= 0 & \text{Given equation} \\
(2x-5)(x+3) &= 0 & \text{Factor } 2x^2 + x - 15 \\
2x - 5 &= 0 \quad \text{or} \quad x + 3 &= 0 & \text{Zero-product property} \\
x &= \frac{5}{2} \quad \text{or} \quad x &= -3 & \text{Solve both linear equations}
\end{align*}
\]

The solutions (roots) of the quadratic equation are \( x = \frac{5}{2} \) and \( x = -3 \). We can check our answers by substituting back into the original equation. We obtain

\[
2 \left( \frac{5}{2} \right)^2 + \frac{5}{2} - 15 = \frac{25}{2} + \frac{5}{2} - 15 = \frac{30}{2} - 15 = 15 - 15 = 0
\]
\[ 2(-3)^2 + (-3) - 15 = 18 - 3 - 15 = 0 \]

**Square-Root Property**

Quadratic equations of the form \((x - p)^2 = q\), where \(p\) and \(q\) are real numbers and \(q > 0\), can be solved easily using the square root property. In the next section, we show that *any* quadratic equation can be put in this form and this is the key to deriving the familiar quadratic formula for solving *any* quadratic equation.

**Square Root Property**

Suppose \(x\) satisfies the quadratic equation \((x - p)^2 = q\), where \(p\) and \(q\) are real numbers and \(q > 0\). Then \(x - p = \sqrt{q}\) or \(x - p = -\sqrt{q}\).

**Example 2.** Solve \(2(x - 5)^2 - 14 = 0\) by using the square root property.

**Solution.** We put \(2(x - 5)^2 - 14 = 0\) in the form \((x - p)^2 = q\) and use the square root property to find its two solutions.

\[
\begin{align*}
2(x - 5)^2 - 14 &= 0 & \text{Given equation} \\
2(x - 5)^2 &= 14 & \text{Add 14} \\
(x - 5)^2 &= 7 & \text{Divide by 2} \\
x - 5 &= \sqrt{7} \quad \text{or} \quad x - 5 = -\sqrt{7} & \text{Square root property} \\
x = 5 + \sqrt{7} \quad \text{or} \quad x = 5 - \sqrt{7} & \text{Solve both linear equations}
\end{align*}
\]

The solutions of the quadratic equation are \(x = 5 + \sqrt{7}\) and \(x = 5 - \sqrt{7}\).

**Quadratic Formula**

Any quadratic equation \(ax^2 + bx + c = 0\) can be solved in terms of its coefficients \(a\), \(b\) and \(c\) using the quadratic formula.

**Quadratic Formula**

The roots of the quadratic equation \(ax^2 + bx + c = 0\), where \(a\), \(b\) and \(c\) are real numbers with \(a \neq 0\), can be found using the quadratic formula

\[
x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}
\]

The discriminant \(D = b^2 - 4ac\) of the quadratic equation \(ax^2 + bx + c = 0\) can be used to determine how many real number solutions this equation has.
The Discriminant

The discriminant of the quadratic equation \( ax^2 + bx + c = 0 \) is \( D = b^2 - 4ac \). The discriminant can be used to determine how many real number solutions the quadratic equation has.

1. If \( D > 0 \), then the equation has two distinct real number solutions.
2. If \( D = 0 \), then the equation has exactly one real number solution.
3. If \( D < 0 \), then the equation has no real number solution.

Example 3. Solve \( x^2 - 6x + 7 = 0 \) using the quadratic formula.

Solution. We identify the coefficients \( a = 1, \ b = -6 \) and \( c = 7 \) to be used in the quadratic formula. We then substitute these numbers into the quadratic formula to obtain

\[
x = \frac{-(6) \pm \sqrt{(-6)^2 - 4(1)(7)}}{2(1)}
\]

\[
= \frac{6 \pm \sqrt{8}}{2}
\]

\[
= \frac{6 \pm 2\sqrt{2}}{2}
\]

\[
= 3 \pm \sqrt{2}
\]

The solutions of the quadratic equation are \( x = 3 + \sqrt{2} \) and \( x = 3 - \sqrt{2} \).

Example 4. Use the discriminant of the given quadratic equation to determine how many real number solutions it has. If it has real solutions, use the quadratic formula to find them.

(a) \( 2x^2 - x - 3 = 0 \)

(b) \( 4x^2 - 12x + 9 = 0 \)

(c) \( x^2 + 2x + 5 = 0 \)

Solution. (a) We identify the coefficients \( a = 2, \ b = -1 \) and \( c = -3 \) to be used in the formula for the discriminant and evaluate it to obtain

\[
D = b^2 - 4ac
\]

\[
= (-1)^2 - 4(2)(-3)
\]

\[
= 25
\]

Since \( D > 0 \), the quadratic equation \( 2x^2 - x - 3 = 0 \) has two solutions and we can find these solutions using the quadratic formula. We obtain

\[
x = \frac{-(1) \pm \sqrt{25}}{2(2)}
\]

\[
x = \frac{1 \pm 5}{4}
\]
The two solutions are \( x = \frac{3}{2} \) and \( x = -1 \).

(b) We identify the coefficients \( a = 4, \ b = -12 \text{ and } c = 9 \) to be used in the formula for the discriminant and evaluate it to obtain

\[
D = b^2 - 4ac \\
D = (-12)^2 - 4(4)(9) \\
D = 0
\]

Since \( D = 0 \), the quadratic equation \( 4x^2 - 12x + 9 = 0 \) has only one solution and we can find it using the quadratic formula. We obtain

\[
x = \frac{-(-12) \pm \sqrt{0}}{2(4)} \\
x = \frac{3}{2}
\]

The solution is \( x = \frac{3}{2} \).

(c) We identify the coefficients \( a = 1, \ b = 2 \text{ and } c = 5 \) to be used in the formula for the discriminant and evaluate it to obtain

\[
D = b^2 - 4ac \\
D = 2^2 - 4(1)(5) \\
D = -16
\]

Since \( D < 0 \), the quadratic equation \( x^2 + 2x + 5 = 0 \) has no real number solutions. If the discriminant \( D \) in the quadratic formula is negative, we cannot take its square root and get a real number. This is the reason the equation has no real number solutions.

The method we use to solve a quadratic equation is often not specified. In this case we can select whatever method works and is most convenient.

**Example 5.** Solve the quadratic equation \( x^2 + 3x = 10 \).

**Solution.** We first place the quadratic equation in form

\[
x^2 + 3x - 10 = 0
\]
It appears that this equation will be easy to factor and try that approach. We obtain
\[
\begin{align*}
\text{Given equation} & : & x^2 + 3x - 10 &= 0 \\
\text{Factor} & : & (x + 5)(x - 2) &= 0 \\
\text{Zero-product property} & : & x + 5 &= 0 \quad \text{or} \quad x - 2 &= 0 \\
\text{Solve both linear equations} & : & x &= -5 \quad \text{or} \quad x &= 2
\end{align*}
\]

The two solutions are \( x = -5 \) and \( x = 2 \).

We could use the quadratic formula and get the same solutions if we choose to do so. We first identify the coefficients \( a = 1, \ b = 3 \) and \( c = -10 \) to be used. We then substitute these numbers into the quadratic formula to obtain
\[
x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}
\]
\[
\begin{align*}
\text{Quadratic formula: } a &= 1, \ b &= 3, \ c &= -10 \\
x &= \frac{-3 \pm \sqrt{3^2 - 4(1)(-10)}}{2(1)} \\
x &= \frac{-3 \pm \sqrt{49}}{2} \\
x &= \frac{-3 \pm 7}{2} \\
x &= \frac{-3 + 7}{2} \quad \text{or} \quad x &= \frac{-3 - 7}{2}
\end{align*}
\]

Use \( + \) to get one solution and \( - \) to get the other.

The two solutions are \( x = 2 \) and \( x = -5 \), confirming what we obtained by factoring.

**Example 6.** Solve the quadratic equation \((x + 2)^2 = 9\).

**Solution.** We have two options in this case. We could square \( x + 2 \) and put the equation in the form in which we could solve it by factoring or by the quadratic formula.

The simpler option is to take advantage of the structure of the equation and use the square root property. We obtain
\[
\begin{align*}
\text{Given equation} & : & (x + 2)^2 &= 9 \\
\text{Square root property; note } \sqrt{9} &= 3 & : & & x + 2 &= \pm 3 \quad & \text{Solve both linear equations} \\
x &= 1 \quad \text{or} \quad x &= -5
\end{align*}
\]

The two solutions are \( x = 1 \) and \( x = -5 \).
Exercise Set 8.1

Solve the quadratic equation by factoring. Solve the same equation by using the quadratic formula to check your answers.

1. \(x^2 - 5x + 6 = 0\) 
2. \(x^2 - 2x = 3\) 
3. \(x^2 + 7x + 12 = 0\) 
4. \(x^2 + 4x + 4 = 0\) 
5. \(x^2 + 5x - 14 = 0\) 
6. \(x^2 = 4x + 12\) 
7. \(2x^2 + 5x - 3 = 0\) 
8. \(3x^2 - x - 2 = 0\)

Use the discriminant of the given quadratic equation to determine how many real number solutions it has. If it has real solutions, use the quadratic formula to find them.

9. \(x^2 + 2x - 5 = 0\) 
10. \(x^2 - 3x = 1\) 
11. \(x^2 + 8 = 0\) 
12. \(x^2 - 6x + 9 = 0\) 
13. \(x^2 + 6x + 1 = 0\) 
14. \(x^2 - 6x + 4 = 0\) 
15. \(2x^2 + 4x + 2 = 0\) 
16. \(2x^2 = 3x + 5\) 
17. \(3x^2 - 2x + 1 = 0\) 
18. \(3x^2 = 6x - 9\)

Solve the given quadratic equation by finding all its real number solutions.

19. \(x^2 - 9 = 0\) 
20. \(x^2 - 5x = 0\) 
21. \(x^2 + x - 6 = 0\) 
22. \(x^2 - 8x + 15 = 0\) 
23. \(x^2 + 11x + 28 = 0\) 
24. \((x - 1)^2 - 4 = 0\) 
25. \(x^2 + 3x + 1 = 0\) 
26. \(2x^2 + x - 3 = 0\) 
27. \(2x^2 - 16 = 0\) 
28. \(4x^2 - 4x - 15 = 0\) 
29. \(\left(x + \frac{1}{2}\right)^2 = 1\) 
30. \(x^2 - 2x + 2 = 0\) 
31. \(x^2 - 4x + 4 = 0\) 
32. \(x^2 - 4x + 2 = 0\)
<table>
<thead>
<tr>
<th></th>
<th>Equation</th>
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<th>Equation</th>
</tr>
</thead>
<tbody>
<tr>
<td>33.</td>
<td>$(x - 5)^2 = 2$</td>
<td>34.</td>
<td>$x^2 + 8 = 0$</td>
</tr>
<tr>
<td>35.</td>
<td>$x^2 = 6x - 9$</td>
<td>36.</td>
<td>$3x^2 - 13x - 10 = 0$</td>
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<tr>
<td>37.</td>
<td>$\left(x - \frac{3}{2}\right)^2 - \frac{1}{4} = 0$</td>
<td>38.</td>
<td>$x^2 + x = 2$</td>
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<tr>
<td>39.</td>
<td>$3x^2 - 5x - 1 = 0$</td>
<td>40.</td>
<td>$2x^2 - x - \frac{1}{2} = 0$</td>
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<td>41.</td>
<td>$4x^2 - x - 5 = 0$</td>
<td>42.</td>
<td>$4(x + 1)^2 = 3$</td>
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<td>43.</td>
<td>$(x + 2)^2 + 4 = 0$</td>
<td>44.</td>
<td>$x^2 - 2x - 35 = 0$</td>
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</table>
8.2. Completing the Square

*KYOTE Standards:* CA 11, CA 17

The technique of *completing the square* can be used to transform *any* quadratic equation to an equivalent equation in the form \((x - p)^2 = q\), where \(p\) and \(q\) are real numbers. We saw in the last section how an equation in this form can be solved using the square root property. This technique is used to derive the quadratic formula.

This technique is also used to write a quadratic function in a form that enables us to sketch its graph without difficulty. We shall see how this is done in the next section.

We begin by highlighting the key idea that is used in completing the square.

**General Approach to Completing the Square**

To complete the square for the quadratic expression \(x^2 + bx\), where \(b\) a real number, we add and subtract \(\left(\frac{b}{2}\right)^2\) to obtain

\[
x^2 + bx = x^2 + bx + \left(\frac{b}{2}\right)^2 - \left(\frac{b}{2}\right)^2
\]

\[
= \left(x + \frac{b}{2}\right)^2 - \frac{b^2}{4}
\]

We can use this approach to complete the square for any quadratic expression and solve any quadratic equation as is shown in the following examples.

**Example 1.** Complete the square for the given quadratic expression.

- **(a)** \(x^2 - 3x\)
- **(b)** \(2x^2 + 4x\)
- **(c)** \(-x^2 + x\)

*Solution.* **(a)** We follow the general approach highlighted above.

\[
x^2 - 3x = \left(x^2 - 3x + \frac{9}{4}\right) - \frac{9}{4}
\]

Given expression

\[
= \left(x - \frac{3}{2}\right)^2 - \frac{9}{4}
\]

Add and subtract \(\left(-\frac{3}{2}\right)^2 = \frac{9}{4}\)

Write \(x^2 - 3x + \frac{9}{4} = \left(x - \frac{3}{2}\right)^2\)

Therefore \(x^2 - 3x = \left(x - \frac{3}{2}\right)^2 - \frac{9}{4}\) after completing the square.
(b) Since the coefficient of $x^2$ in the expression $2x^2 + 4x$ is not 1, we cannot complete the square we did in part (a). Instead, we first factor out 2 and then follow the general approach highlighted above.

\[
2x^2 + 4x = 2(x^2 + 2x)
\]

\[
= 2(x^2 + 2x + 1 - 1)
\]

\[
= 2((x + 1)^2 - 1)
\]

\[
= 2(x + 1)^2 - 2
\]

Therefore $2x^2 + 4x = 2(x + 1)^2 - 2$ after completing the square.

(c) Since the coefficient of $x^2$ in the expression $-x^2 + x$ is not 1, we first factor out $-1$ and then follow the general approach highlighted above.

\[
-x^2 + x = -(x^2 - x)
\]

\[
= -(x^2 - x + \frac{1}{4} - \frac{1}{4})
\]

\[
= -(x - \frac{1}{2})^2 - \frac{1}{4}
\]

\[
= -(x - \frac{1}{2})^2 + \frac{1}{4}
\]

Therefore $-x^2 + x = -(x - \frac{1}{2})^2 + \frac{1}{4}$ after completing the square.

The approach we take to solve a quadratic equation by completing the square is essentially the same as the approach we take to complete the square for a quadratic expression. But there are some minor differences that are highlighted in Example 2 that arise because we are dealing with equations rather than expressions.

**Example 2.** Solve the given quadratic equation by completing the square.

(a) $x^2 + 6x - 7 = 0$

(b) $x^2 - 2x - 1 = 0$

(c) $4x^2 - 12x + 5 = 0$

**Solution.** (a) We solve $x^2 + 6x - 7 = 0$ by completing the square as follows:

\[
x^2 + 6x - 7 = 0
\]

\[
x^2 + 6x = 7
\]

Add 7
\[ x^2 + 6x + 9 = 7 + 9 \quad \text{Add} \left( \frac{6}{2} \right)^2 = 9 \]

Note that we added 9 to both sides of the equation instead of adding and subtracting 9 from \( x^2 + 6x \) as we would do when completing the square for the quadratic expression \( x^2 + 6x \).

\[ (x+3)^2 = 16 \quad \text{Write} \quad x^2 + 6x + 9 = (x+3)^2 \]

| \( x+3 = 4 \) | or | \( x+3 = -4 \) | Square root property |
| \( x = 1 \) | or | \( x = -7 \) | Solve both linear equations |

The solutions of \( x^2 + 6x - 7 = 0 \) are \( x = 1 \) and \( x = -7 \).

(b) We solve \( x^2 - 2x - 1 = 0 \) by completing the square as follows:

\[ x^2 - 2x - 1 = 0 \quad \text{Given equation} \]
\[ x^2 - 2x = 1 \quad \text{Add 1} \]
\[ x^2 - 2x + 1 = 1 + 1 \quad \text{Add} \left( -\frac{2}{2} \right)^2 = 1 \]

Note that we added 1 to both sides of the equation instead of adding and subtracting 1 from \( x^2 - 2x \) as we would do when completing the square for the quadratic expression \( x^2 - 2x \).

\[ (x-1)^2 = 2 \quad \text{Write} \quad x^2 - 2x + 1 = (x-1)^2 \]

| \( x-1 = \sqrt{2} \) | or | \( x-1 = -\sqrt{2} \) | Square root property |
| \( x = 1 + \sqrt{2} \) | or | \( x = 1 - \sqrt{2} \) | Solve both linear equations |

The solutions of \( x^2 - 2x - 1 = 0 \) are \( x = 1 + \sqrt{2} \) and \( x = 1 - \sqrt{2} \).

(c) We solve \( 4x^2 - 12x + 5 = 0 \) by completing the square as follows:

\[ 4x^2 - 12x + 5 = 0 \quad \text{Given equation} \]
\[ 4x^2 - 12x = -5 \quad \text{Subtract 5} \]
\[ x^2 - 3x = -\frac{5}{4} \quad \text{Divide by 4} \]

Note that we divide both sides of the equation by 4 instead of factoring out 4 as we would do when completing the square for the quadratic expression \( 4x^2 - 12x \).

\[ x^2 - 3x + \frac{9}{4} = -\frac{5}{4} + \frac{9}{4} \quad \text{Add} \left( -\frac{3}{2} \right)^2 = \frac{9}{4} \]
\[ \left( x - \frac{3}{2} \right)^2 = 1 \quad \text{Write} \quad x^2 - 3x + \frac{9}{4} = \left( x - \frac{3}{2} \right)^2 \]
\[ x - \frac{3}{2} = 1 \quad \text{or} \quad x - \frac{3}{2} = -1 \quad \text{Square root property} \]

\[ x = \frac{5}{2} \quad \text{or} \quad x = \frac{1}{2} \quad \text{Solve both linear equations} \]

The solutions of \( 4x^2 - 12x + 5 = 0 \) are \( x = \frac{5}{2} \) and \( x = \frac{1}{2} \).

---

**Exercise Set 8.2**

*Complete the square for the given quadratic expression*

1. \( x^2 + 2x \)  
2. \( x^2 - 4x \)  
3. \( x^2 + 5x \)  
4. \( x^2 - 7x \)  
5. \( x^2 - x \)  
6. \( 2x^2 + 12x \)  
7. \( 3x^2 - 12x \)  
8. \( 3x^2 + 9x \)  
9. \( 4x^2 + 24x \)  
10. \( 2x^2 + x \)  
11. \( 2x^2 - 5x \)  
12. \( 3x^2 - 2x \)  
13. \( -x^2 + 4x \)  
14. \( -2x^2 + 3x \)

*Solve the given quadratic equation by completing the square*

15. \( x^2 + 2x = 0 \)  
16. \( x^2 - x = 0 \)  
17. \( x^2 - 2x - 3 = 0 \)  
18. \( x^2 - 4x + 3 = 0 \)  
19. \( x^2 + 6x + 4 = 0 \)  
20. \( 2x^2 - 4x - 1 = 0 \)  
21. \( 4x^2 - 4x - 3 = 0 \)  
22. \( x^2 - 4x - 2 = 0 \)  
23. \( 2x^2 + 8x + 1 = 0 \)  
24. \( 3x^2 - 6x - 1 = 0 \)  
25. \( x^2 + 4x - 1 = 0 \)  
26. \( 4x^2 - 3x - 4 = 0 \)
27. \( x^2 + 2x + 5 = 0 \)

29. \( 4x^2 - 4x + 1 = 0 \)

31. \( x^2 - 2x + 3 = 0 \)

28. \( (x + 3)(x - 1) = 1 \)

30. \( (x - 2)(x + 4) = -7 \)

32. \( x^2 + 5x + 5 = 0 \)
8.3. Graph Quadratic Functions

KYOTE Standards: CA 17

In the previous section, we saw how the technique of completing the square for a quadratic expression could be used to solve a quadratic equation. We use this technique in this section to sketch the graph a quadratic function.

A quadratic function associates with each real number $x$ exactly one number $y$ given by $y = ax^2 + bx + c$, where $a \neq 0$, $b$ and $c$ are real numbers. Each such pair of numbers $(x, y)$ forms a graph in the plane called a parabola. This parabola is in general difficult to sketch. But it can be can be easily sketched if we can put the quadratic function in vertex form by completing the square.

**Definition 1.** The vertex form of a quadratic function is

$$y = m(x - h)^2 + k$$

where $m \neq 0$, $h$ and $k$ are real numbers.

Suppose a quadratic function is in vertex form $y = m(x - h)^2 + k$. Its graph is easy to construct. Its vertex is at the point $(h, k)$ and it is symmetric around the line $x = h$. It opens upward if $m > 0$ and has a minimum value $k$ at $x = h$. It opens downward if $m < 0$ and has a maximum value $k$ at $x = h$. The examples show how this works.

**Example 1.** Consider the quadratic function $y = 2(x - 1)^2 - 4$ in vertex form.

(a) Find its vertex and its minimum value.

(b) Find its $x$- and $y$-intercepts.

(c) Sketch its graph. Place the coordinates of the points corresponding to the vertex and the intercepts on the graph.

**Solution.** (a) The vertex of $y = 2(x - 1)^2 - 4$ is $(1, -4)$. Since $2(x - 1)^2$ is 0 when $x = 1$ and positive when $x \neq 1$, we see that the graph of the function is a parabola that has a minimum value of $-4$ when $x = 1$. We also see that it is symmetric about the line $x = 1$ and opens upward. We plot some points to help visualize these properties although you are not required to do this in the exercises.

<table>
<thead>
<tr>
<th>$x$</th>
<th>$y = 2(x - 1)^2 - 4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>-1</td>
<td>4</td>
</tr>
<tr>
<td>0</td>
<td>-2</td>
</tr>
<tr>
<td>1</td>
<td>-4</td>
</tr>
<tr>
<td>2</td>
<td>-2</td>
</tr>
<tr>
<td>3</td>
<td>4</td>
</tr>
</tbody>
</table>
(b) We set $x = 0$ in $y = 2(x-1)^2 - 4$ to obtain $y = -2$. Thus the $y$-intercept is $-2$ and $(0, -2)$ is a point on the parabola, as is confirmed in the table above.

We set $y = 0$ in $y = 2(x-1)^2 - 4$ and solve the equation $2(x-1)^2 - 4 = 0$ for $x$ to find the $x$-intercepts using the square root property.

\[
\begin{align*}
2(x-1)^2 - 4 &= 0 & \text{Given equation} \\
2(x-1)^2 &= 4 & \text{Add 4} \\
(x-1)^2 &= 2 & \text{Divide by 2} \\
x-1 &= \sqrt{2} \quad \text{or} \quad x-1 = -\sqrt{2} & \text{Square root property} \\
x &= 1 + \sqrt{2} \quad \text{or} \quad x = 1 - \sqrt{2} & \text{Solve both linear equations}
\end{align*}
\]

The $x$-intercepts are $x = 1 + \sqrt{2}$ and $x = 1 - \sqrt{2}$, and the points $(1 + \sqrt{2}, 0)$ and $(1 - \sqrt{2}, 0)$ are points on the parabola.

(c) The sketch of the graph, with the coordinates of the points corresponding to the vertex and the intercepts on the graph, is shown below.

\[\text{Example 2. Consider the quadratic function } \ y = -x^2 + 4x + 5.\]

(a) Write $y = -x^2 + 4x + 5$ in vertex form by completing the square.

(b) Find its vertex and its maximum value.

(c) Find its $x$- and $y$-intercepts.

(d) Sketch its graph. Place the coordinates of the points corresponding to the vertex and the intercepts on the graph.

\textbf{Solution. (a)} We write $y = -x^2 + 4x + 5$ in vertex form by first completing the square for the quadratic expression $-x^2 + 4x$ as we did in the previous section. We obtain
\[-x^2 + 4x\] is the given expression.

Factor out \(-1\):

\[-(x^2 - 4x)\]

Add and subtract \((4/2)^2 = 4\):

\[-(x^2 - 4x + 4 - 4)\]

Write \(x^2 - 4x + 4 = (x - 2)^2\):

\[-(x - 2)^2 + 4\]

We replace \(-x^2 + 4x\) by \(-(x - 2)^2 + 4\) in \(y = -x^2 + 4x + 5\) to obtain:

\[
y = -(x - 2)^2 + 4 + 5 = -(x - 2)^2 + 9
\]

The vertex form of \(y = -x^2 + 4x + 5\) is therefore \(y = -(x - 2)^2 + 9\).

\[\textbf{(b)}\] The vertex of \(y = -(x - 2)^2 + 9\) is \((2, 9)\). Since \(-(x - 2)^2\) is 0 when \(x = 2\) and negative when \(x \neq 2\), we see that the graph of the function has a maximum value of 9 when \(x = 2\). We also see it is a parabola symmetric about the line \(x = 2\) that opens downward. You can plot some points as in Example 1 to visualize these properties.

\[\textbf{(c)}\] We set \(x = 0\) in either one of the two forms of the function, \(y = -x^2 + 4x + 5\) or \(y = -(x - 2)^2 + 9\). We see in either case that the \(y\)-intercept is 5 and that \((0, 5)\) is a point on the parabola.

We set \(y = 0\) in either one of the two forms of the function, \(y = -x^2 + 4x + 5\) or \(y = -(x - 2)^2 + 9\). We then solve either one of the quadratic equations, \(-x^2 + 4x + 5 = 0\) or \(-(x - 2)^2 + 9 = 0\), to find the \(x\)-intercepts.

The equation in vertex form is usually easier to solve since we can apply the square root property. But it is instructive in our example to solve both equations and confirm that the answers in each case are the same.

\[
\begin{align*}
-x^2 + 4x + 5 &= 0 & \text{Given equation} \\
x^2 - 4x - 5 &= 0 & \text{Multiply by } -1 \\
(x + 1)(x - 5) &= 0 & \text{Factor} \\
x + 1 &= 0 & \text{or} & \quad x - 5 &= 0 & \text{Zero-product property} \\
x &= -1 & \text{or} & \quad x &= 5 & \text{Solve both linear equations}
\end{align*}
\]
The solutions are $x = -1$ and $x = 5$.

We confirm this answer by solving $-(x-2)^2 + 9 = 0$.

Given equation

\[-(x-2)^2 + 9 = 0\]

\[(x-2)^2 = 9\] Subtract 9; multiply by $-1$

\[x-2 = 3 \quad \text{or} \quad x-2 = -3\] Square root property

\[x = 5 \quad \text{or} \quad x = -1\] Solve both linear equations

The solutions are confirmed to be $x = -1$ and $x = 5$. Thus the $x$-intercepts are $-1$ and $5$, and the points $(-1,0)$ and $(5,0)$ are points on the parabola.

(d) The sketch of the graph, with the coordinates of the points corresponding to the vertex and the intercepts on the graph, is shown below.

Exercise Set 8.3

A quadratic function in vertex form is given in exercises 1-12. In each case:

(a) Find its vertex and its maximum or minimum value.

(b) Find its $x$- and $y$-intercepts.

(c) Sketch its graph. Place the coordinates of the points corresponding to the vertex and the intercepts on the graph.

1. $y = (x+2)^2 - 9$
2. $y = (x-5)^2 - 4$
3. $y = -(x+2)^2 + 1$
4. $y = 2(x-3)^2 - 8$
5. $y = 2(x+1)^2 + 4$
6. $y = 3(x-2)^2 - 6$
7. $y = -2(x+1)^2 + 5$
8. $y = -(x-4)^2 - 1$
9. \[ y = -\frac{1}{2} \left( x + \frac{3}{2} \right)^2 + \frac{15}{8} \]

10. \[ y = \left( x - \frac{5}{2} \right)^2 - \frac{9}{4} \]

11. \[ y = 4(x+1)^2 - 8 \]

12. \[ y = \frac{1}{2} \left( x - \frac{3}{2} \right)^2 - \frac{25}{8} \]

A quadratic function is given in exercises 13-24. In each case:

(a) Write its equation in vertex form by completing the square.
(b) Find its vertex and its maximum or minimum value.
(c) Find its x- and y-intercepts.
(d) Sketch its graph. Place the coordinates of the points corresponding to the vertex and the intercepts on the graph.

13. \[ y = x^2 + 4x \]

14. \[ y = x^2 - 2x - 3 \]

15. \[ y = -x^2 + 10x \]

16. \[ y = x^2 + 4x + 3 \]

17. \[ y = 3x^2 - 6x - 1 \]

18. \[ y = x^2 + x - 1 \]

19. \[ y = -x^2 + 4x - 1 \]

20. \[ y = -2x^2 + 8x - 6 \]

21. \[ y = -x^2 + 4x - 5 \]

22. \[ y = 2x^2 + 6x \]

23. \[ y = x^2 + 2x - 2 \]

24. \[ y = 2x^2 + 4x + 3 \]
8.4. Applications of Quadratic Equations

**KYOTE Standards: CA 14**

Solving applied problems with quadratic equations involves the same five-step approach as solving applied problems with linear equations that we considered in Section 6.3. We state this five-step approach with “linear” replaced by “quadratic.”

**Five Steps to Use as Guidelines in Solving Applied Problems with Equations**

1. **Define the Variable.** Read the problem carefully. The problem asks you to find some quantity or quantities. Choose one of these quantities as your variable and denote it by a letter, often the letter \( x \). Write out a clear description of what the quantity \( x \) represents.

2. **Express All Other Unknown Quantities in Terms of the Variable.** Read the problem again. There are generally unknown quantities in the problem other than the one represented by the variable, say \( x \). Express these unknown quantities in terms of \( x \).

3. **Set up the Equation.** Set up a quadratic equation that gives a relationship between the variable and the unknown quantities identified in Step 2.

4. **Solve the Equation.** Solve the quadratic equation you obtain.

5. **Interpret Your Answer.** Write a sentence that answers the question posed in the problem. **Caution.** The variable name \( x \) or other unknown quantities expressed in terms of \( x \) should not appear in your interpretation.

**Example 1.** A rectangular room 3 feet longer than it is wide is to be carpeted with carpet costing $2.50 per square foot. The total cost of the carpet is $675. What are the length and width of the room?

**Solution.** We are asked to find the length and the width of the room. So we let

\[
 x = \text{width of the room in feet} \quad \text{Step 1}
\]

The length of the room in terms of \( x \) can be written

\[
 x + 3 = \text{length of the room in feet} \quad \text{Step 2}
\]

Since the total cost of the carpet depends on the area of the room, we also write the area of the room in terms of \( x \).

\[
 x(x + 3) = \text{area of the room in square feet} \quad \text{Step 2}
\]

We can find the total cost of the carpet by multiplying the area of the room in square feet by the cost of the carpet per square foot. The total cost is therefore
The relationship between the area of the room and the cost of carpeting it can be written as an equation.

\[ 2.50 \frac{\text{dollars}}{\text{ft}^2} \times x(x+3) \text{ ft}^2 = 2.50x(x+3) \text{ dollars} \]

We solve this quadratic equation. We first simplify it by dividing both sides by 2.50 to obtain

\[ x(x+3) = 270 \]

Note that we could also have obtained this equation by finding the area of the rectangle by dividing the total cost 675 by the cost 2.50 per square foot.

\[ \frac{675 \text{ dollars}}{2.50 \frac{\text{dollars}}{\text{ft}^2}} = 270 \text{ ft}^2 \]

We solve this equation using the quadratic formula after identifying the needed coefficients.

\[ \begin{align*}
  x(x+3) &= 270 & \text{Given equation} \\
  x^2 + 3x &= 270 & \text{Expand} \\
  x^2 + 3x - 270 &= 0 & \text{Subtract 270}
\end{align*} \]

The coefficients of \( x^2 + 3x - 270 = 0 \) are \( a = 1 \), \( b = 3 \) and \( c = -270 \). We substitute these numbers into the quadratic formula to obtain

\[ x = \frac{-3 \pm \sqrt{3^2 - 4(1)(-270)}}{2(1)} \quad \text{Quadratic formula: } a = 1, \ b = 3, \ c = -270 \]

\[ x = \frac{-3 \pm \sqrt{1089}}{2} \quad \text{Simplify} \]

\[ x = \frac{-3 \pm 33}{2} \quad \text{Simplify } \sqrt{1089} = 33 \]

The solutions are \( x = 15 \) and \( x = -18 \).
Note that we could also have solved the equation \( x^2 + 3x - 270 = 0 \) by factoring to obtain \((x - 15)(x + 18) = 0\), thus giving the same solutions \(x = 15\) and \(x = -18\). But factoring in general becomes increasingly difficult as the coefficients of the quadratic function become larger.

The interpretation is particularly important in this problem. We reject the solution \(x = -18\) because we cannot have a width of \(-18\) feet! Thus the relevant solution in this problem is \(x = 15\), which is the width of the room in feet. The length of the room is \(x + 3 = 18\), which is also measured in feet.

We interpret the answer.

\[ \text{The length of the room is 18 feet and the width is 15 feet.} \]

**Exercise Set 8.4**

**KYOTE Standards:** CA 14

1. The sum of the squares of two consecutive odd integers is 74. Find the two integers.

2. The sum of squares of two consecutive integers is 85. Find the two integers.

3. Two numbers have a sum of 21 and a product of 104. Find the numbers.

4. The length of a rectangular garden is twice its width and its area is 578 square feet. What are the length and width of the garden?

5. One leg of a right triangle is three times longer than its other leg. What are the lengths of the two legs if the hypotenuse is \(7\sqrt{10}\) inches long?

6. The hypotenuse of a right triangle is twice as long as one of its legs and the other leg has length \(9\sqrt{3}\) centimeters. What is the length of the hypotenuse?

7. One leg of a right triangle is 3 times longer than its other leg. What are the lengths of the two legs and the hypotenuse if the area of the triangle is 96 square inches?

8. A rectangle is 10 meters longer than it is wide and its area is 875 square meters. What are the length and width of the rectangle?

9. A rectangle of length 8 inches and width 5 inches is cut from a square piece of cardboard. If the area of the remaining cardboard is 321 square inches, what is the length of the square piece of cardboard?
10. The length of a rectangle is 2 feet longer than it is wide and its area is 224 square feet. What is its perimeter?

11. A rectangle is 24 feet long. The length of the diagonal between opposite corners of the rectangle is 12 feet more than its width. What is its width?

12. A rectangle has a perimeter of 160 centimeters and an area of 1500 square centimeters. What are the length and width of the rectangle?

13. The length of the diagonal between opposite corners of a rectangle is 20 inches and its length is 4 inches longer than its width. What is the perimeter of the rectangle?

14. The length of the diagonal between opposite corners of a rectangle is twice its width and its length is 27 feet. What is the area of the rectangle?

15. A fence costing $20 per foot is purchased to enclose a rectangular field whose length is 4 feet longer than its width and whose area is 437 square feet. What is the total cost of the fence?

16. A sail is in the form of a right triangle with the vertical leg 4 feet longer than the horizontal leg and with hypotenuse $4\sqrt{13}$ feet. What is the cost of the material used to make the sail if this material costs $10 per square foot?